Equivalence of Many-Gluon Green Functions in Duffin-Kemmer-Petieu and Klein-Gordon-Fock Statistical Quantum Field Theories

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Abstract

We prove the equivalence of many-gluon Green functions in Duffin-Kemmer-Petieu (DKP) and Klein-Gordon-Fock (KGF) statistical quantum field theories. The proof is based on the functional integral formulation for the statistical generating functional in a finite-temperature quantum field theory. As an illustration, we calculate one-loop polarization operators in both theories and show that their expressions indeed coincide.

Keywords: statistical quantum field theory, gluon Green functions, path integral, renormalization, equivalence.

1 Introduction

This work is a straightforward generalization of the articles [1]–[3] which established the equivalence of many-photon Green functions in DKP and KGF statistical quantum field theories.

In Section 2 we present a general proof of equivalence using the functional integral method in statistical quantum field theory. From the physical

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viewpoint, our result is understandable qualitatively: in non-zero temperature conditions gluons do not become massive (and thus do not acquire any chemical potential), so that their intrinsic properties remain the same. To illustrate this, in Section 3 we calculate one-loop polarization operators in both theories and show that these operators actually coincide. Section 4 contains conclusive remarks.

2 Coincidence of Many-Gluon Green Functions in DKP and KGF Theories for Finite Temperatures

To construct a generating functional $Z(J, \bar{J}, J_\mu)$ for the Green functions (GF) in statistical theory, we should perform transition to Euclidean space and then limit the integration area along x_4: $0 \le x_4 \le \beta$, where $\beta = 1/T$, T is the temperature, and $J, \bar{J}, J_\mu$ are the external currents. For simplicity, from now on we restrict ourselves to the case of *fundamental* representation of the *SU(N)* group (see [4], [5]).

In DKP theory, the functional integral describing interaction between the gluon field A_μ^a and charged particles of spin-0 and mass m has the following form (in the α-gauge):

$$
Z_{\mathrm{DKP}}(J^i, \bar{J}^j, J_\mu) = Z_0 \int_\beta DA_\mu^a D\psi^i D\bar{\psi}^i \exp\left\{\left[-\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2\right.\right.
$$
$$
\left.\left. - \bar{c}^i(\partial_\mu D_\mu^{ij} c^j) + \bar{\psi}^i\left(i\beta_\mu D_\mu^{ij} - m\delta^{ij}\right)\psi^j + \bar{J}^i\psi^i + J^i\bar{\psi}^i + A_\mu^a J_\mu^a\right]d^4x\right\}. \quad (1)
$$

Here $a = 1, 2, \ldots N^2 - 1$ is the group index; $i, j = 1, 2, \ldots N_j$,

$$
D_\mu^{ij} = \delta^{ij}\partial_\mu - ig\left(A_\mu^a t^a\right)^{ij}
$$
$$
[t_a, t_b]_- = if_{abc}t_c, \quad (2)
$$

where f_{abc} are the *SU(N)* group structure constants.

The Euclidean β_μ-matrices in Eq.(1) are chosen as in [3] (see Eq.(4)); the fields $c^i, \bar{c}^j$ are the Faddeev-Popov ghosts [6]. As for the functional integrations in this formula, they are understood as

$$
\int_\beta DA_\mu^a = \int \prod_{0 \le x_4 \le \beta} \prod_{-\infty \le x_i \le +\infty} \prod_{a=1}^{N^2-1} \prod_\mu dA_\mu^a. \quad (3)
$$

To prove the coincidence of many-gluon Green functions, let us integrate out the ψ^i and $\bar{\psi}^j$ fields in Eq.(1). We get:

$$Z_{\text{DKP}}(J^i, \bar{J}^j, J_\mu) = Z_0 \int_\beta DA_\mu^a \exp\left\{-\int_\beta d^4x \left[\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2\right.\right.$$

$$\left. - \bar{c}^i(\partial_\mu D_\mu^{ij} c^j) + J_\mu^a A_\mu^a + \text{Tr}\,\ln S^{ii}(x,x,A_\mu^a)\right]$$

$$\left. - \int_\beta d^4x\, d^4y\, \bar{J}^i(x) S^{ij}(x,y,A^a) J^j(y)\right\}, \tag{4}$$

where
$$S^{ij}(x,y,A) = \left(\beta_\mu D_\mu^{ij} - m\delta^{ij}\right)^{-1}\delta(x-y) \tag{5}$$

is the Green function for the DKP-particle in the external Yang-Mills field A_μ^a. The term $\text{Tr}\,\ln S(x,x,A)$ gives rise to all vacuum diagrams in perturbation theory.

On the other hand,

$$\exp \text{Tr}\,\ln S^{ij}(x,x,A^a) = \det S^{ij}(x,y,A^a)$$

$$= \int_\beta D\psi^i D\bar{\psi}^j \exp\left\{-\int_\beta d^4x\, \bar{\psi}^i(x)\left(\beta_\mu D_\mu^{ij} + \delta^{ij}m\right)\psi^j(x)\right\}, \tag{6}$$

where $\psi^i(x)$ is the column vector

$$\psi^i(x) = \begin{pmatrix} \phi^i(x) \\ \partial_4\phi^i(x) \\ \partial_1\phi^i(x) \\ \partial_2\phi^i(x) \\ \partial_3\phi^i(x) \end{pmatrix}, \tag{7}$$

and thus we can rewrite Eq.(6) in the component form as:

$$\exp \text{Tr}\,\ln S^{ij}(x,x,A^a) = \det S^{ij}(x,y,A^a)$$

$$= \int_\beta \prod_{\mu=1}^4 D_\mu\phi^{*i} D_\mu\phi^j D\phi^{*i} D\phi^j \exp\left\{-\int_\beta d^4x \left[\phi^{*i} D_\mu^{ij}\phi_\mu^j\right.\right.$$

$$\left.\left. + \phi_\mu^{*i} D_\mu^{ij}\phi^j + m\left(\phi^i\phi^{*i} + \phi_\mu^{*i}\phi_\mu^i\right)\right]\right\}. \tag{8}$$

After integrating over ϕ_μ^{*i} and ϕ_μ^i we obtain

$$\det S^{ij}(x,y,A^a) \equiv \det G^{ij}(x,y,A) = \exp \text{Tr}\,\ln G^{ii}(x,x,A)$$

$$= \int_\beta D\phi^i D\phi^{*j} \exp\left\{+\frac{1}{m}\int_\beta d^4x \phi^{*i}\left((D_\mu^2)^{ij} - m^2\delta^{ij}\right)\phi^j\right\}, \tag{9}$$

where
$$G^{ij}(x,y,A^a) = \left(-(D_\mu^2)^{ij} + m^2\delta^{ij}\right)\delta^4(x-y) \tag{10}$$

is the Green function of the KGF equation in the external field $A_\mu^a(x)$.

It follows from Eqs.(8)–(10) that many-gluon Green functions coincide in DKP and KGF theories. This completes the proof of equivalence of many-gluon Green functions in these theories.

3 Polarization Operator in One-Loop Approximation

In order to prove the equality of one-loop polarization operators in DKP and KGF statistical theories it is sufficient to consider the loops formed by scalar massive particles, since all other one-loop diagrams coincide in these theories.

The one-loop polarization operator in DKP theory has the following form:

$$\Pi_{\mu\nu}^{\text{DKP}}(k) = \frac{g^2}{(2\pi)^2\beta}\text{Tr}\int d\mathbf{p}\beta_\mu(t^a)^{ij}S^{jk}(p+k)\beta_\nu(t^a)^{kl}S^{li}(p), \qquad (11)$$

where

$$S^{jk}(p) = \delta^{jk}(i\hat{p} - m)^{-1}. \qquad (12)$$

One easily checks that

$$S^{jk} = \delta^{jk}(i\hat{p} - m)^{-1} = -\frac{\delta^{ik}}{m}\left(\frac{i\hat{p}(i\hat{p} + m)}{p^2 + m^2} + 1\right), \qquad (13)$$

$$\hat{p} = \beta_\mu p_\mu,\ p^2 = p_4^2 + \mathbf{p}^2,\ p_4 = \frac{2\pi n}{\beta},\ -\infty < n < +\infty,$$

$$(i\hat{p} - m)^{ij}S^{jk}(\hat{p}) = \delta^{ik}. \qquad (14)$$

Using Eq.(11)–(14) we obtain the polarization operator in DKP theory (in g^2-approximation):

$$\Pi_{\mu\nu}^{\text{DKP}}(k) = \frac{g^2}{(2\pi)^2\beta}(t_a)^{ij}(t_a)^{ji}\sum_{p_4}\int d\mathbf{p}\Big(\frac{(2p+k)_\mu(2p+k)_\nu}{(p^2+m^2)((p+k)^2+m^2)}$$

$$-\frac{\delta_{\mu\nu}}{p^2+m^2} - \frac{\delta_{\mu\nu}}{(p+k)^2+m^2}\Big). \qquad (15)$$

In KGF theory, the one-loop polarization operator equals to:[1]

$$\Pi_{\mu\nu}^{\text{KGF}}(k) = \frac{g^2(t^a)^{ij}(t^a)^{ji}}{(2\pi)^3\beta}\sum_{p_4}\int d\mathbf{p}\left(\frac{(2p+k)_\mu(2p+k)_\nu}{(p^2+m^2)((p+k)^2+m^2)} - \frac{2\delta^{\mu\nu}}{p^2+m^2}\right). \qquad (16)$$

The term proportional to $\delta_{\mu\nu}$ in Eq.(16) plays an important role in the proof of transversality of $\Pi_{\mu\nu}$ (i.e., $k_\mu\Pi_{\mu\nu} = 0$). This term appears because of the term $\sim (A_\mu^a(x))^2$ entering $(D_\mu^2)^{ij}$ in Eq.(10).

After the substitution $(p + k) \to p$ in the term $\delta_{\mu\nu}((p + k)^2 + m^2)^{-1}$ of Eq.(15) it changes into $\delta_{\mu\nu}(p^2+m^2)^{-1}$, so that the right-hand sides of Eq.(15) and Eq.(16) coincide, and become formally gauge-invariant. This coincidence

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[1]The expressions (15) and (16) coincide with Eqs.(14) and (10) in [3] up to the substitution $e^2 \to g^2(t_a)^{ij}(t_a)^{ji}$.

of $\Pi_{\mu\nu}^{\text{DKP}}$ and $\Pi_{\mu\nu}^{\text{KGF}}$ in one-loop approximation confirms the general proof of equivalence presented in Section 2.

In relativistic quantum field theory, the $\Pi_{\mu\nu}(k)$ tensor has the form:

$$\Pi_{\mu\nu} = \left(k_\mu k_\nu - k^2 \delta_{\mu\nu}\right) \Pi(k^2). \tag{17}$$

In quantum statistics, $\Pi_{\mu\nu}$ depends on the two vectors: k_μ and u_μ, the latter being the unit vector of the media velocity. Thus, the most general expression reads (see [7], page 75)

$$\Pi_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)A_1 + \left(u_\mu u_\nu - \frac{k_\mu u_\nu (ku)}{k^2}\right.$$
$$\left. - \frac{k_\nu u_\mu (ku)}{k^2} + \frac{k_\mu k_\nu (ku)^2}{k^2}\right)A_2 \equiv \Phi_{\mu\nu}^1 A_1 + \Phi_{\mu\nu}^2 A_2. \tag{18}$$

Introducing the notation (in any approximation)

$$a_1 \equiv \Pi_{\mu\mu} = 3A_1 + \lambda A_2$$
$$a_2 = u_\mu \Pi_{\mu\nu} u_\nu = \lambda(A_1 + \lambda A_2), \quad \lambda = 1 - \frac{(ku)^2}{k^2}, \tag{19}$$

we have

$$A_1 = \frac{1}{2}\left(a_1 - \frac{1}{\lambda}a_2\right), \quad A_2 = \frac{1}{2\lambda}\left(-a_1 + \frac{3}{\lambda}a_2\right). \tag{20}$$

If the system is at rest[2]

$$\lambda = 1 - \frac{k_4^2}{k^2} \tag{21}$$

and

$$a_2 = \left(1 - \frac{k_4^2}{k^2}\right)A_1 + \left(1 - \frac{k_4^2}{k^2}\right)^2 A_2. \tag{22}$$

To proceed further, it is convenient to represent a_1 and a_2 in the form

$$a_i = a_i^R + a_i^\beta, \quad i = 1, 2.$$

Here the terms a_i^R do not depend on β and must be renormalized; a_i^β depend on β and must vanish when $\beta \to \infty$:

$$\lim_{\beta \to \infty} a_i^\beta = 0. \tag{23}$$

Now Eq.(18) may be rewritten in the following form:

$$\Pi_{\mu\nu} = \frac{1}{2}\Phi_{\mu\nu}^1\left(a_1^R - \frac{1}{\lambda}a_2^R + a_1^\beta - \frac{1}{\lambda}a_2^\beta\right) + \frac{1}{2\lambda}\Phi_{\mu\nu}^2\left(-a_1^R + \frac{3}{\lambda}a_2^R - a_1^\beta + \frac{3}{\lambda}a_2^\beta\right). \tag{24}$$

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[2]One can show that the representation (18) is, strictly speaking, valid only if the system is at rest: $\mathbf{u} = 0, u_4 = 1$ (see [7], Chapter 11, Section 7).

The terms $\sim \Phi_{\mu\nu}^2$ should vanish when $\beta \to \infty$. Correspondingly, in this limit we get the $\Pi_{\mu\nu}$ tensor of Euclidean quantum field theory. Since a_1^R and a_2^R do not depend on β, we obtain after renormalization:

$$a_2^R = \frac{\lambda}{3} a_1^R. \tag{25}$$

Thus

$$\lim_{\beta \to \infty} \Pi_{\mu\nu} = \frac{1}{3}\Phi_{\mu\nu}^1 a_1^R, \quad \text{or } \Pi_{\mu\nu} = a_1^R. \tag{26}$$

Let us calculate a_1 and a_2 using the general formula for summation over p_4 in the r.h.s. of Eq.(15). We may drop the terms proportional to $\delta_{\mu\nu}$ in r.h.s. of Eqs.(10) and (14), because such terms vanish after regularization and renormalization. Then

$$a_1 = -\frac{g^2(N^2 - 1)}{(2\pi)^3 2\beta} \sum_{p_4} \int d\mathbf{p} \frac{(2p + k)^2}{(p^2 + m^2)((p+k)^2 + m^2)}, \tag{27}$$

$$a_2 = -\frac{g^2(N^2 - 1)}{(2\pi)^3 2\beta} \sum_{p_4} \int d\mathbf{p} \frac{(2p + k)_4^2}{(p^2 + m^2)((p+k)^2 + m^2)}, \tag{28}$$

on account of (see [5], p.114, Eq.(7.32))

$$(t^a)^{ij}(t^a)^{ji} = \frac{N^2 - 1}{2}. \tag{29}$$

The general formula for summation over p_4 reads (see [7], p.123, Appendix 3, and [8], p.299):

$$\frac{1}{\beta} \sum_n f(\frac{2\pi n}{\beta}, K) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} d\omega f(\omega, K) + \frac{1}{2\pi} \int\limits_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\omega \frac{f(\omega, K) + f(-\omega, K)}{e^{-i\beta\omega} - 1}. \tag{30}$$

After juxtaposition of Eqs.(27),(28) with Eqs.(27),(28) for statistical QED [3] and substitution of $g^2(N^2 - 1)/2$ for e^2 we obtain corresponding expressions for statistical quantum gluodynamics.

Eq.(36) in [3] is replaced by

$$\lim_{\beta \to \infty} \Pi_{\mu\nu} = \left(-\frac{k_\mu k_\nu}{k^2} + \delta_{\mu\nu}\right) \left(\frac{g^2(N^2 - 1)}{96\pi^2}\right) k^4 \int\limits_{4m^2}^{\infty} dz \frac{(1 - 4m^2/z^2)^{3/2}}{z^2(z^2 + k^2)}. \tag{31}$$

The same expression for $\Pi_{\mu\nu}$ also follows from Eq.(11) found in [9], where the photon Green function was calculated using the dispersion approach.

From Eq.(25) and (35) from [??] we find the expression for a_2^R:

$$a_2^R = \frac{\lambda}{3} a_1^R = \frac{g^2(N^2 - 1)k^2}{96\pi^2}(k^4 - k^2) \int\limits_{4m^2}^{\infty} dz \frac{(1 - 4m^2/z^2)^{3/2}}{z^2(z^2 + k^2)}. \tag{32}$$

Finally, we get for a_1^β and a_2^β ($\mu \neq 0$):

$$
a_1^\beta = \frac{g^2(N^2-1)}{32\pi^2}(4m^2 + k^2) \int\limits_0^\infty \frac{p\,dp}{E|\mathbf{k}|} \left(e^{\beta(E-\mu)} - 1\right)^{-1}
$$
$$
\times \ln \frac{(k^2 + 2p|\mathbf{k}|)^2 + 4E^2 k_4^2}{(k^2 - 2p|\mathbf{k}|)^2 + 4E^2 k_4^2} \tag{33}
$$
$$
a_2^\beta = \frac{g^2(N^2-1)}{32\pi^2} \int\limits_0^\infty \frac{p^2\,dp}{Ep|\mathbf{k}|} \left(e^{\beta(E-\mu)} - 1\right)^{-1} \left\{ (E^2 - k_4^2) \right.
$$
$$
\left. \times \ln \frac{(k^2 + 2p|\mathbf{k}|)^2 + 4E^2 k_4^2}{(k^2 - 2p|\mathbf{k}|)^2 + 4E^2 k_4^2} + 2iEk_4 \ln \frac{(k^2 + 2iEk_4)^2 - 4p^2\mathbf{k}^2}{(k^2 - 2iEk_4)^2 - 4p^2\mathbf{k}^2} \right\}, \tag{34}
$$

where $E = (p^2 + m^2)^{1/2}$.

4 Conclusions

We have proven the equivalence of many-gluon Green functions in DKP and KGF statistical field theories (Section 2) and calculated one-loop polarization operators to illustrate this equivalence (Section 3).

Thus, the series of our works [1–3,9] prove that both theories lead to identical results for *observable* physical quantities. In this respect, it would be interesting to prove the equivalence of the results related to the processes which involve unstable particles and, in particular, to apply the methods of DKP theory to the Standard Model.

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